THREE NOTIONS OF TROPICAL RANK FOR SYMMETRIC MATRICES

DUSTIN CARTWRIGHT AND MELODY CHAN

Abstract. We introduce and study three different notions of tropical rank for symmetric and dissimilarity matrices in terms of minimal decompositions into rank 1 symmetric matrices, star tree matrices, and tree matrices. Our results provide a close study of the tropical secant sets of certain nice tropical varieties, including the tropical Grassmannian. In particular, we determine the dimension of each secant set, the convex hull of the variety, and in most cases, the smallest secant set which is equal to the convex hull.

1. Introduction

In this paper, we study tropical secant sets and rank for symmetric matrices. Our setting is the *tropical semiring* $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, where tropical addition is given by $x \oplus y = \min(x, y)$ and tropical multiplication is given by $x \odot y = x + y$. The kth *tropical secant set* of a subset V of $\mathbb{R}^N$ is defined to be the set of points

$$\{x \in \mathbb{R}^N : x = v_1 \oplus \cdots \oplus v_k,\ v_i \in V\},$$

where $\oplus$ denotes coordinate-wise minimum. This set is typically not a tropical variety and thus we prefer the term "secant set" to "secant variety," which has appeared previously in the literature. The *rank* of a point $x \in \mathbb{R}^N$ with respect to V is the smallest integer k such that x lies in the kth tropical secant set of V, or ∞ if there is no such k.

In [7], Develin, Santos, and Sturmfels define the Barvinok rank of a matrix, not necessarily symmetric, to be the rank with respect to the subset of $n \times n$ rank 1 matrices, and their definition serves as a model for ours. In addition, they define two other notions of rank, Kapranov rank and tropical rank, for which there are no analogues in this paper. Further examination of ranks of not necessarily symmetric matrices can be found in the review article [1].

We give a careful examination of secant sets and rank with respect to three families of tropical varieties in the space of symmetric matrices and the space of dissimilarity matrices. By a $n \times n$ *dissimilarity matrix* we simply mean a function from $\binom{[n]}{2}$ to $\mathbb{R}$, which we will write as a symmetric matrix without any entries on the diagonal. There is a natural projection from $n \times n$ symmetric matrices to $n \times n$ dissimilarity matrices which we denote by π. For example,

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \pi(M) = \begin{bmatrix} * & 1 & 0 & 0 \\ 1 & * & 0 & 0 \\ 0 & 0 & * & 1 \\ 0 & 0 & 1 & * \end{bmatrix} \tag{1}$$

are a symmetric matrix and dissimilarity matrix respectively.

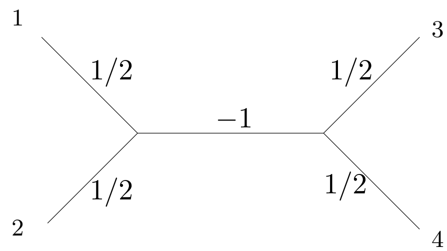

FIGURE 1. Weighted tree whose distance matrix is $\pi(M)$ from (1).

Our first family is the *tropical Veronese* of degree 2, which is the tropicalization of the classical space of symmetric matrices of rank 1. It is a classical linear subspace of the space of symmetric matrices consisting of those matrices which can be written as $v^T \odot v$ for some row vector v. The rank of a matrix with respect to the tropical Veronese is called *symmetric Barvinok rank*, because it is the symmetric analogue of Barvinok rank.

Second, we consider the *space of star trees*, which is the image of the tropical Veronese under the projection π. Equivalently, it can be obtained by first projecting the classical Veronese onto its off-diagonal entries and then tropicalizing. The classical variety and its secant varieties were studied in [9]. The tropical variety is a classical linear subspace of the space of dissimilarity matrices, and we call the corresponding notion of rank *star tree rank*. The name reflects the fact that the matrices with star tree rank 1 are precisely those points of the tropical Grassmannian which correspond to trees with no internal edges, i.e. star trees, in in the identification below.

Third, we consider the *tropical Grassmannian* $G_{2,n}$, which is the tropicalization of the Grassmannian of 2-dimensional subspaces in an n-dimensional vector space, and was first studied in [14]. It consists of exactly those dissimilarity matrices arising as the distance matrix of a weighted tree with n leaves in which internal edges have negative weights. Therefore, we call the points in the tropical Grassmannian *tree matrices*, and call rank with respect to the tropical Grassmannian the *tree rank*. Note that our definition of tree rank differs from that in [12, Ch. 3], which uses a different notion of mixtures.

Our first two families are examples of classical linear spaces, whose secant sets were studied by Mike Develin [6]. He defines a natural polyhedral fan such that the tropical secant set is the support of this polyhedral fan. Moreover, Theorem 2.1 in [6] gives an algorithm for computing the rank of a point with respect to a fixed linear space. In contrast, we do not know of a good algorithm for computing the rank with respect to the tropical Grassmannian (see Section 9), and we do not know of a natural fan structure.

We use our examples of M and $\pi(M)$ from (1) to illustrate our three notions of rank. Proposition 5 tells us that the symmetric Barvinok rank of M is 4. Theorem 10 tells us that the star tree rank of $\pi(M)$ is 2. Explicitly, we have

$$\pi(M) = \begin{bmatrix} * & 1 & 0 & 0 \\ 1 & * & 2 & 2 \\ 0 & 2 & * & 1 \\ 0 & 2 & 1 & * \end{bmatrix} \oplus \begin{bmatrix} * & 1 & 2 & 2 \\ 1 & * & 0 & 0 \\ 2 & 0 & * & 1 \\ 2 & 0 & 1 & * \end{bmatrix}.$$

Finally, the tree rank of $\pi(M)$ is 1 by Proposition 15, which can also be seen explicitly from the weighted tree in Figure 1. This example shows that all three of our notions of rank can be different.

However, for any $n \times n$ symmetric matrix M, we have

$$\text{symmetric Barvinok rank}(M) \geq \text{star tree rank}(\pi(M)) \geq \text{tree rank}(\pi(M)). \tag{2}$$

The first inequality follows from the fact that the set of dissimilarity matrices of star tree rank 1 is the projection of the set of matrices of symmetric Barvinok rank 1. The second inequality follows from the fact that the space of star trees is contained in the tropical Grassmannian.

The rest of the paper is organized as follows. In Section 2, we present a technique for proving lower bounds on rank. We introduce a graph associated to a matrix for each of our notions of rank; the chromatic number of this graph is a lower bound on the rank of the matrix. The same technique applies to provide a lower bound to the rank of any point with respect to any tropical prevariety, although in general it may produce a hypergraph instead of a graph.

We examine symmetric Barvinok rank, star tree rank, and tree rank in Sections 3, 4, and 5 respectively. We prove upper bounds on the rank in each case, and with the exception of tree rank, our upper bounds are sharp. We show that the symmetric Barvinok rank of an $n \times n$ symmetric matrix can be infinite, but even when the rank is finite it can exceed n, and in fact can grow quadratically in n (Theorem 6). For each notion of rank, the set of matrices with rank at most k is a union of polyhedral cones, and we compute the dimension of these sets, defined as the dimension of the largest cone. In each case, the dimension of the tropical secant set equals the dimension of the clasical secant variety, confirming Draisma's observation that tropical geometry provides useful lower bounds for the dimensions of classical secant varieties [8]. Finally, we give a combinatorial characterization of each notion of rank for a 0/1 matrix in terms of graph covers.

In Section 6, we examine 3×3 symmetric matrices and explicitly characterize the stratification by symmetric Barvinok rank. In Sections 7 and 8, we do the same for the 5×5 dissimilarity matrices and the stratifications by star tree rank and tree rank respectively. In particular, we show that the lower bounds from the chromatic number in Section 2 are exact in these cases. We close with some open problems in Section 9.

2. Lower bounds on rank via hypergraph coloring

Before we examine our three notions of rank, we give a general combinatorial construction: a hypergraph whose chromatic number yields a lower bound on rank.

Recall that a *hypergraph* consists of a ground set, called vertices, and a set of subsets of the ground set, called hyperedges. The *chromatic number* of a hypergraph H, denoted $\chi(H)$, is the smallest number r such that the vertices of H can be partitioned into r color classes with no hyperedge of H monochromatic. In particular, if H contains a hyperedge of size 1, then $\chi(H)$ is ∞.

Now, suppose we have a tropical prevariety $V \subseteq \mathbb{R}^N$. Recall that a tropical polynomial

$$p(x_1, \ldots, x_N) = \bigoplus_{i=1}^{t} a_i \odot x_1^{c_{i1}} \odot \cdots \odot x_N^{c_{iN}} \tag{3}$$

defines a tropical hypersurface consisting of those vectors $x \in \mathbb{R}^N$ such that the minimum in evaluating $p(x)$ is achieved at least twice. A *tropical prevariety* is the intersection of finitely many tropical hypersurfaces, and any finite set S of tropical polynomials defining the prevariety V is called a *tropical basis*.

Now, given a point $w \in \mathbb{R}^N$ and a tropical basis S for V, we construct a hypergraph on ground set $[N]$ as follows. Let p from (3) be a tropical polynomial in S, with all exponents $c_{ij} \geq 0$. If the minimum is achieved uniquely when p is evaluated at w, then we add a hyperedge E whose elements correspond to the coordinates that appear with non-zero exponent in the unique minimal term. The *deficiency hypergraph* of w with respect to V and S consists of hyperedges coming from all polynomials in S with a unique minimum at w. In particular, the deficiency hypergraph has no hyperedges (and thus has chromatic number 1) if and only if w is in V.

Proposition 1. *If H is the deficiency hypergraph constructed above, then the rank of $w \in \mathbb{R}^N$ with respect to $V \subseteq \mathbb{R}^N$ is at least $\chi(H)$.*

Proof. Suppose that w has rank r with respect to V, and let $w = v_1 \oplus \cdots \oplus v_r$ be an expression of w as the tropical sum of r points $v_i \in V$. We will construct an r-coloring of the deficiency hypegraph H, which will show that $\chi(H) \leq r$. For each $i \in [N]$, there is at least one v_j which agrees with w in the ith coordinate, so arbitrarily pick one such j as the color for vertex i in H. Let E be a hyperedge of H and p the associated tropical polynomial. We claim that E cannot be monochromatic with color j. Each coordinate of v_j is greater than or equal to the corresponding coordinate of w, so each term of $p(v_j)$ is greater than or equal to the corresponding term of $p(w)$. On the other hand, the minimum in the evaluation of $p(v_j)$ is achieved at least twice, so the minimum must be strictly greater than $p(w)$. Thus, v_j cannot agree with w for all coordinates in E, so E is not monochromatic of color j. This holds for any color j, so we have constructed an r-coloring, and thus, $\chi(H) \leq r$. □

Corollary 2. *If the deficiency hypergraph H has a hyperedge of size* 1*, then the rank of w with respect to V is infinite.*

Proof. If H has a hyperedge of size 1, then $\chi(H)$ is ∞, and thus the rank of w is infinite. □

We do not know of any examples in which this lower bound is actually strict; see Section 9.

For the varieties considered in this paper, we will take quadratic tropical bases and thus the deficiency hypergraph will always be a graph (possibly with loops). Accordingly, we will call it the *deficiency graph*.

3. Symmetric Barvinok rank

Recall from the introduction that the symmetric Barvinok rank of a symmetric matrix M is the smallest number r such that M can be written as the sum of r rank 1 symmetric matrices. The 2×2 minors $x_{ij}x_{kl} \oplus x_{il}x_{kj}$ of M for $i \neq k$ and $l \neq j$ form a tropical basis for the variety of rank 1 symmetric matrices. We will always construct our deficiency graph with respect to this tropical basis.

Our first observation is that the symmetric Barvinok rank of a matrix can be infinite. More precisely,

Proposition 3. *If M is a symmetric matrix and $2M_{ij} < M_{ii} + M_{jj}$ for some i and j, then the symmetric Barvinok rank of M is infinite.*

Proof. The tropical polynomial $x_{ij}^2 \oplus x_{ii}x_{jj}$ is in the tropical basis, so if $2M_{ij} < M_{ii} + M_{jj}$ for some i and j, then the deficiency graph for M has a loop at the node ij. Therefore, M has infinite rank by Corollary 2. □

In fact, the converse to Proposition 3 is also true; see Theorem 6. In order to construct decompositions into rank 1 matrices, we need the following lemma.

Lemma 4. *Let M be an $m \times m$ symmetric Barvinok rank 1 matrix, $n > m$ an integer and C any real number. Then there exists an $n \times n$ symmetric rank 1 matrix N such that the upper left $m \times m$ submatrix is M and every other entry is at least C.*

Proof. Since M has rank 1, then $M = v^T \odot v$ for some row vector v. Let $C' = \max\{\frac{1}{2}C, C - v_i\}$, and let w be the vector consisting of v followed by C' repeated $n - m$ times. Then $N = w^T \odot w$ has the desired properties. □

Remark 1. We will use the symbol ∞ in an entire row and column of a matrix to denote sufficiently large values that maintain the property of being rank 1. So, if M is an $m \times m$ rank 1 matrix, then

$$\begin{bmatrix} M & \infty \\ \infty & \infty \end{bmatrix}$$

denotes the $(m+1) \times (m+1)$ matrix obtained by applying Lemma 4 with $n = m+1$. The value of C will be clear from the context.

Next, we give a graph-theoretic characterization of the symmetric Barvinok rank of 0/1-matrices. We define a *clique cover* of a simple graph G to be a collection of r complete subgraphs such that every edge and every vertex of G is in some element of the collection. Given an $n \times n$ symmetric 0/1 matrix M with zeroes on the diagonal, define G_M to be the graph whose vertices are the integers $[n]$, and which has an edge between i and j if and only if $M_{ij} = 0$.

Proposition 5. *Suppose M is a symmetric $0/1$ matrix with zeroes on the diagonal. Then the symmetric Barvinok rank of M is the size of a smallest clique cover of G_M.*

On the other hand, suppose that M is a symmetric $0/1$ matrix with at least one entry of 1 on the diagonal. If there exist i and j such that $M_{ii} = 1$ and $M_{ij} = 0$, then the symmetric Barvinok rank of M is infinite. Otherwise, let M' be the maximal principal submatrix with zeroes on the diagonal. The symmetric Barvinok rank of M is one greater than the symmetric Barvionk rank of M'.

Proof. First suppose that all diagonal entries of M are 0. Let $G_1, \ldots, G_r$ be a clique cover of G_M. Let v_i be the 0/1 row vector whose jth entry is 0 if j is a vertex of G_i, and let $M_i = v_i^T \odot v_i$. Then we claim that $M = \bigoplus M_i$. Each 0 entry in M corresponds to an edge or vertex of G_M which is in some G_i, so there is a 0 in M_i. For $j \neq k$ such that $M_{jk} = 1$, some G_i contains the vertex j, and hence misses the vertex k, so $(M_i)_{jk} = 1$. On the other hand, the entry is at least 1 in the other rank 1 matrices because none of the corresponding graphs contains jk as an edge. Thus the rank of M is at most r.

Conversely, suppose N is a rank 1 symmetric matrix in a decomposition of M. Since all entries of M are non-negative, so are the entries of N, so the rank 1

condition says $N_{ii} = N_{jj} = 0$ if and only if $N_{ij} = 0$. By the "if" direction, we can define a graph G_N whose vertices and edges correspond to the diagonal and off-diagonal zeroes of N respectively. By the "only if" direction, this is a complete graph. Every position with a 0 in M must be 0 for some rank 1 matrix in the decomposition, so the graphs G_N form a clique cover of G_M as N ranges over all rank 1 symmetric matrices in the decomposition. Thus, the rank of M is exactly r.

Next, we suppose M has at least one 1 on the diagonal and let $M' \subsetneq M$ be as in the statement. If there exist i and j such that $M_{ii} = 1$ but $M_{ij} = 0$, then the rank of M is infinite by Proposition 3. Otherwise, we claim that the rank of M is $r+1$. Extending each rank 1 summand of a minimal decomposition of M' by Lemma 4 and adding in the all ones matrix shows that the rank of M is at most $r+1$. On the other hand, it is straightforward to check that a rank 1 summand containing a 1 entry on the diagonal can contain no zeroes, so does not contribute to a clique cover for $G_{M'}$. So the rank of M is exactly $r+1$ in this case. □

Remark 2. This characterization gives us two families of matrices which have rank n and $\lfloor n^2/4 \rfloor$ respectively, namely those corresponding to the trivial graph with n isolated vertices and the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. In the latter case, $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is triangle-free, so no clique can consist of more than one edge. On the other hand, there are $\lfloor n^2/4 \rfloor$ edges in the graph, so $\lfloor n^2/4 \rfloor$ cliques are needed in a cover. In fact, these two examples have the maximum possible rank for $n \times n$ matrices, as shown below.

Theorem 6. *Suppose that M is a symmetric $n \times n$ matrix with $M_{ii} + M_{jj} \leq 2M_{ij}$ for all i and j. Then the symmetric Barvinok rank of M is at most* $\max\{n, \lfloor n^2/4 \rfloor\}$, *and this bound is tight. Thus, every matrix with finite rank has rank at most* $\max\{n, \lfloor n^2/4 \rfloor\}$.

Proof. Subtracting $M_{ii}/2$ from the ith row and column for all i does not change the rank, so we can assume that the diagonal entries of M are 0 and hence, by hypothesis, the off-diagonal entries are non-negative.

The statement is trivial for $n = 1$. For $n = 2$, we have

$$M = \begin{bmatrix} 0 & M_{12} \\ M_{12} & 2M_{12} \end{bmatrix} \oplus \begin{bmatrix} 2M_{12} & M_{12} \\ M_{12} & 0 \end{bmatrix}. \tag{4}$$

For $n = 3$, we assume, without loss of generality, that $M_{12} \geq M_{23}$. Then

$$M = \begin{bmatrix} 0 & M_{12} & M_{13} \\ M_{12} & 2M_{12} & M_{12}+M_{13} \\ M_{13} & M_{12}+M_{13} & 2M_{13} \end{bmatrix} \oplus \begin{bmatrix} \infty & \infty & \infty \\ \infty & 0 & M_{23} \\ \infty & M_{23} & 2M_{23} \end{bmatrix} \oplus \begin{bmatrix} \infty & \infty & \infty \\ \infty & \infty & \infty \\ \infty & \infty & 0 \end{bmatrix}, \tag{5}$$

where "∞" is as in Remark 1. For n at least 4, the proof is by induction, using the following lemma:

Lemma 7. *Let $n \geq 4$. Suppose that for any $(n-2) \times (n-2)$ matrix N of finite rank, there exists a matrix N' of rank at most $\lfloor (n-2)^2/4 \rfloor$, such that N' is identical to N except possibly in one diagonal entry, where the entry of N' is greater than or equal to the entry of N. Then any $n \times n$ matrix has rank at most $\lfloor n^2/4 \rfloor$.*

Remark 3. The exceptional diagonal entry in the hypothesis makes the statement of the lemma slightly stronger than what is required for an inductive proof of an $\lfloor n^2/4 \rfloor$ upper bound. However, we will use the lemma to establish the base cases

of the induction, $n = 4$ and $n = 5$, which require a weaker hypothesis because for $n = 2$ and $n = 3$ the $\lfloor n^2/4 \rfloor$ upper bound on rank does not hold.

Proof of Lemma 7. Let M be an $n \times n$ matrix with finite rank. Without loss of generality, assume that the entry M_{12} is minimal among all off-diagonal elements. We apply the hypothesis to the principal submatrix indexed by $\{3, \dots, n\}$. Applying Lemma 4, we have a collection of at most $\lfloor (n-2)^2/4 \rfloor$ rank 1 matrices whose tropical sum agrees with M except for the first two rows and columns and possibly one diagonal entry, which, without loss of generality, we assume to be M_{44}. For each $4 \leq i \leq n$, take the rank 1 matrix which has arbitrary large values except for the $\{1, 2, i\}$ principal submatrix, which is:

$$\begin{bmatrix} 2M_{1i} & M_{1i} + M_{2i} & M_{1i} \\ M_{1i} + M_{2i} & 2M_{2i} & M_{2i} \\ M_{1i} & M_{2i} & 0 \end{bmatrix}.$$

Note that since M_{12} was chosen to be minimal, we have that $M_{12} \leq M_{1i} + M_{2i}$.

Finally, switching indices 1 and 2 if necessary, we can assume that $M_{13} \geq M_{23}$, and hence $M_{12} + M_{13} \geq M_{23}$. Then, we take two matrices which are "∞" outside of the $\{1, 2, 3\}$ principal matrices, which are, respectively,

$$\begin{bmatrix} 0 & M_{12} & M_{13} \\ M_{12} & 2M_{12} & M_{12} + M_{13} \\ M_{13} & M_{12} + M_{13} & 2M_{13} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \infty & \infty & \infty \\ \infty & 0 & M_{23} \\ \infty & M_{23} & 2M_{23} \end{bmatrix},$$

recalling the meaning of "∞" from Remark 1.

This yields a decomposition of M into at most $\lfloor (n-2)^2/4 \rfloor + (n-3) + 2 = \lfloor n^2/4 \rfloor$ symmetric rank 1 matrices. □

To complete the proof of the Theorem 6, we note that taking all but the last term of (4) and (5) gives the hypothesis to Lemma 7 for $n = 4$ and 5. The desired upper bound follows by induction, which consists of applying Lemma 7 with $N' = N$. Finally, Remark 2 shows that this bound is tight. □

Theorem 8. *The dimension of the space of symmetric $n \times n$ matrices of symmetric Barvinok rank at most r is $\binom{n+1}{2} - \binom{n-r+1}{2}$, which is the dimension of the classical secant variety, i.e. the space of classical symmetric matrices of classical rank at most r.*

Proof. Let $D = \binom{n+1}{2} - \binom{n-r+1}{2}$. The tropical secant variety is contained in the tropicalization of the classical secant variety, so the dimension is at most D, by the Bieri-Groves Theorem [2, Thm. A]. Thus, it is sufficient to find an open neighborhood in which the tropical variety has dimension D. For i from 1 to r, let $v_i = (C, \dots, C, v_{i,i}, \dots, v_{i,n})$ be a vector with C for the first $i - 1$ entries. Choose the coordinates $v_{i+1,j}$ to be smaller than all the $v_{i,j}$ and let C be very large. Then,

$$v_1^T \odot v_1 \oplus \cdots \oplus v_r^T \odot v_r = \begin{bmatrix} 2v_{11} & v_{11} + v_{12} & \cdots & v_{11} + v_{1n} \\ v_{11} + v_{12} & 2v_{22} & \cdots & v_{22} + v_{2n} \\ \vdots & \vdots & & \vdots \\ v_{11} + v_{1n} & v_{22} + v_{2n} & \cdots & 2v_{rn} \end{bmatrix}$$

This matrix is an injective function of the vector entries v_{ij} for $i \leq r$ and $j \geq i$. Thus, it defines a neighborhood of the rth secant set of the desired dimension

$$n + (n-1) + \cdots + (n-r+1) = \binom{n+1}{2} - \binom{n-r+1}{2} = D. \qquad \square$$

4. Star tree rank

Recall from the introduction that a star tree matrix is one which can be written as $\pi(v^T \odot v)$ for $v \in \mathbb{R}^n$ a row vector. The star tree matrices from a classical linear space in the space of $n \times n$ dissimilarity matrices defined by the tropical polynomials

$$x_{ij}x_{kl} \oplus x_{ik}x_{jl} \qquad \text{for } i,\ j,\ k, \text{ and } l \text{ distinct integers.} \tag{6}$$

In this section, the deficiency graph will always be taken with respect to this tropical basis. The following lemma is an immediate consequence of Lemma 4.

Lemma 9. *Let M be an $m \times m$ star tree matrix, and let $n > m$ be any integer and C any real number. Then there exists an $n \times n$ star tree matrix with M as the upper left $m \times m$ submatrix and all other entries greater than C.*

Unlike the case of symmetric Barvinok rank, the star tree rank is always finite.

Theorem 10. *For n at least 3, the star tree rank of a $n \times n$ dissimilarity matrix M is at most $n-2$, and this bound is sharp. In particular, the dissimilarity matrix defined by $M_{ij} = \min\{i, j\}$ has star tree rank $n-2$.*

Proof. The proof of the upper bound is by induction on n. For $n = 3$, the equations in (6) are trivial, and thus every 3×3 dissimilarity matrix is a star tree matrix. For $n > 3$, let M be an $n \times n$ dissimilarity matrix, and denote by M' the upper left $(n-1) \times (n-1)$ submatrix. By the inductive hypothesis, we can write M' as the tropical sum of $n-3$ star tree matrices. We can extend each of these to $n \times n$ star tree matrices by Lemma 9, and their tropical sum will agree with M except in the last column and row. Let w be the vector defined by $w_i = M_{in} + C$ for $i < n$ and $w_n = -C$, for C a sufficiently large number. Then the tropical sum of the previous $n-3$ matrices together with $\pi(w^T \odot w)$ equals M.

Now let M be the dissimilarity matrix defined by $M_{ij} = \min\{i, j\}$ as in the statement. We claim that the deficiency graph of M has chromatic number $n-2$. For every $i < j < k < l$, we have

$$M_{ik} + M_{jl} = M_{il} + M_{jk} = i + j < M_{ij} + M_{kl} = i + k, \tag{7}$$

so the deficiency graph has an edge between ik and jl, and an edge between il and jk. We refer to these types of edges as "overlapping" and "nesting" respectively.

We prove that the deficiency graph of M has chromatic number at least $n-2$ by induction on n. The case of $n = 3$ is clear. Let n be greater than 3 and fix a coloring of the deficiency graph of M. Let S be the set of nodes of the same color c as the node $1n$. There is a "nesting" edge between $1n$ and every node other than those of the form $1i$ or in for some i. Thus, every node in S is either of the form $1i$ or in. Furthermore, because of the "overlapping" edges, there must be an integer m such that if $1i$ is in S, then $i \leq m$ and if jn is in S, then $m \leq j$. Now consider the set of $n-3$ nodes consisting of $1i$ for $m < i < n$ and jn for $1 < j < m$. By our construction of m, none of them is in S. Therefore, if they have distinct colors, then the coloring has at least $n-2$ colors, which is what we wanted to show.

Otherwise, two of these nodes have the same color, and by the "overlapping" edges and symmetry we can assume that they are $1i$ and $1i'$, with $m < i < i'$, which have color $c' \neq c$. Any node jk with $j < k$ and $1 < j < i$ will share an edge with one of these two nodes, so cannot have color c'. On the other hand, if the node $1l$ with $l \leq m$ is adjacent to jk with $j < k$, then we must have $1 < j < l$. But $l \leq m < i$, so jk cannot have color c'. Thus, we can assume that all of the nodes $1l$ with $l \leq m$ also have color c' without changing the fact that the coloring is proper. With this change, the only nodes with color c are of the form jn. By restricting to the nodes with coordinates less than n, we have a coloring of the deficiency graph of the $(n-1) \times (n-1)$ matrix without the color c, so by the inductive hypothesis we're done. □

Remark 4. The matrix with maximal star tree rank in the previous theorem is in fact in the Grassmannian, i.e. it has tree rank 1. Indeed, from (7), we see that the four-point condition holds. Alternatively, M arises as the distance matrix of the following weighted tree. Let T be the caterpillar tree with n internal vertices, connected in order by edges of weight $-1/2$. The ith leaf vertex is connected to the ith internal vertex by an edge of weight $i/2$. For $i < j$, the distance from leaf i to leaf j is $i/2 + (j-i)(-1/2) + j/2 = i$, which is equal to the corresponding entry in the matrix M from Theorem 10. In order to make a proper phylogenetic tree, we should remove the first and last internal vertices and combine the adjacent edge weights.

Next, we give a graph theoretic characterization of the star tree rank of 0/1-matrices. For M a 0/1 dissimilarity matrix, we define G_M to be the graph whose edges correspond to the zeroes of M. As in the case of symmetric Barvinok rank, we can characterize the star tree rank of M in terms of covers of G_M, this time by both cliques and star trees. We will also say that a cover of G_M by cliques and star trees is a *solid cover* if for every pair of distinct vertices i and j either:

(1) there is an edge between i and j,
(2) either i or j belongs to a clique in the cover,
(3) either i or j is the center of a star tree in the cover, or
(4) both i and j are leaves of the same star tree.

Proposition 11. *Let M be a 0/1 dissimilarity matrix. Let r be the minimal number of graphs in a cover of G_M by cliques and star trees, such that every edge (but not necessarily every vertex) is in some element of the cover. Then M has star tree rank either r or $r+1$.*

Moreover, if G_M has a solid cover by r graphs, then M has star tree rank r.

Proof. Let $G_1, \ldots, G_r$ be a cover of G_M by cliques and star trees. For G_i a clique, define v_i to be the 0/1 vector whose 0 entries correspond to the vertices of the clique. If G_i is a star tree consisting of a central vertex c_i and edges to vertices in the set I_i, then define v_i to be the row vector which is $-1/2$ in the c_i entry, $1/2$ for the entries corresponding to I_i and $3/2$ otherwise. In either case, define $M_i = \pi(v_i^T \odot v_i)$. Then M_i has 0 entries corresponding to the edges of G_i. Thus, the tropical sum of the M_i has the same 0 entries as M. Moreover, if the cover is a solid cover then the tropical sum is equal to M, so M has rank at most r. Otherwise, some of the positive entries of the tropical sum are greater than 1. By additionally taking the tropical sum with the all ones matrix, we see that M has rank at most $r+1$.

Conversely, suppose that $M_i = \pi(v_i^T \odot v_i)$ is a term in a representation of M as the tropical sum of star tree matrices. Then we claim that the zeroes of M_i correspond to either a star tree or a complete graph. If all entries of v_i are non-negative, then the zeroes of M_i correspond to the complete graph on the vertices where v_i is 0. Otherwise, since M_i must be non-negative, there can be at most one negative entry of v_i, say with value $-a$; then all other entries must be at least a. Then the 0 entries of M_i correspond to the star tree with edges between the entry with $-a$ and the entries with value a. Thus, any decomposition of M as the tropical sum of star tree metrics yields a cover of G_M by cliques and star trees, so M has tropical rank at least r. □

We do not know if the definition of a solid cover can be weakened in any way. In other words, we do not know of any 0/1 matrices M such that r is the minimal size of a cover of G_M by cliques and star trees, and G_M does not have a solid cover of size r, but M has star tree rank r.

In contrast to symmetric Barvinok rank, the upper bound of $n-2$ on the star tree rank of an $n \times n$ dissimilarity matrix cannot be achieved by a 0/1 matrix for large n. Recall that the *Ramsey number* $R(k, k)$ is the smallest integer such that any graph on at least $R(k, k)$ vertices has either a clique or a independent set of size k. Then we have the following stronger bound on the star tree rank of a 0/1 matrix.

Proposition 12. *For $n \geq R(k, k)$, any $n \times n$ 0/1 dissimilarity matrix has star tree rank at most $n - k + 1$.*

Proof. By the assumption on n, the graph G_M has either a clique of size k or an independent set of size k. In the former case, we can cover G_M by a star tree centered at each vertex not part of the clique, together with the clique itself. This gives a solid cover by $n - k + 1$ subgraphs. In the latter case, we can just take the star trees centered at the vertices not in the independent set, giving a cover of G_M by $n - k$ subgraphs. In either case, Proposition 11 shows that M has rank at most $n - k + 1$. □

Corollary 13. *For $n \geq 18$, every $n \times n$ 0/1 dissimilarity matrix has star tree rank at most $n - 3$.*

Proof. The Ramsey number $R(4, 4)$ is 18 [13]. □

Theorem 14. *Let r and n be positive integers. Then the dimension of the space of dissimilarity $n \times n$ matrices of star tree rank at most r is*

$$\min\left\{\binom{n+1}{2} - \binom{n-r+1}{2}, \binom{n}{2}\right\}.$$

Proof. Let $D = \min\left\{\binom{n+1}{2} - \binom{n-r+1}{2}, \binom{n}{2}\right\}$ be the dimension from the theorem statement. The dimension cannot be any larger than D by the Bieri-Groves Theorem [2, Thm. A], because D is the dimension of the classical secant variety, according to Theorem 2 in [9]. Therefore, it is sufficient to construct a matrix with star tree rank r which has a D-dimensional neighborhood of star tree rank r matrices. If $r \geq n$, then D is $\binom{n}{2}$, the dimension of the set of $n \times n$ dissimilarity matrices. Since higher secant sets cannot have smaller dimension, it is sufficient to assume $r \leq n$.

We will construct r vectors $v_1, \ldots, v_r$, with $v_{k,i}$ denoting the ith entry of v_k, and then define M to be the tropical sum of the star tree matrices $\pi(v_k^T \odot v_k)$. First,

we fix any order on the set of pairs of distinct integers $S = \{(i,j) : r < i < j \leq n\}$. Then, for $1 \leq k \leq r$ and $r < i \leq n$, we choose $v_{k,i}$ as follows: if the kth pair of integers includes i, then we choose $v_{k,i}$ in the range $0 < v_{k,i} < 1$ and otherwise we choose $v_{k,i} > 2$. For i in the range $k \leq i \leq r$, we choose $v_{k,i}$ inductively beginning with $k = r$. We choose $v_{r,r}$ arbitrarily. Then for $k < r$, and $k \leq i \leq r$, we choose $v_{k,i}$ to be much greater than any of the $v_{k+1,j}$ already chosen. Finally, let C be a large real number and set $v_{k,i}$ equal to C, for $i < k$. Let M be the tropical sum of the $M_k := \pi(v_k^T \odot v_k)$.

We claim that the set of matrices which can be gotten in this way forms a D-dimensional affine linear neighborhood. For $i \leq r$ and $i < j$, the (i,j) entry of M comes from M_i and in particular, is equal to $v_{i,i} + v_{i,j}$. For a fixed i, and taking $j > i$, these entries give us $n - i$ linearly independent functions on the matrix M. Moreover, if (i,j) is the kth pair in the ordering on S, and $k \leq r$, then the (i,j) entry of M comes from M_k and is equal to $v_{k,i} + v_{k,j}$. These are linearly independent from each other and from all of the previous functions. Since the size of S is $\binom{n-r}{2}$, the number of linearly independent functions on the matrix M is

$$\begin{aligned}(n-1) + (n-2) + \cdots + (n-r) &+ \min\left\{r, \binom{n-r}{2}\right\} \\ &= \binom{n}{2} - \binom{n-r}{2} + \min\left\{r, \binom{n-r}{2}\right\} \\ &= \min\left\{\binom{n+1}{2} - \binom{n-r+1}{2}, \binom{n}{2}\right\},\end{aligned}$$

which is the desired dimension, D. $\square$

Remark 5. In fact, the difficult part of Theorem 2 in [9] is proving the lower bound on the dimension of the classical secant variety. Our computation of the dimension of the tropical secant variety provides an alternative proof of this lower bound.

5. Tree rank

The tropical Grasmmannian $G_{2,n}$ is the tropical variety defined by the 3-term Plücker relations:

$$x_{ij}x_{k\ell} \oplus x_{ik}x_{j\ell} \oplus x_{i\ell}x_{jk} \qquad \text{for all } i < j < k < \ell. \tag{8}$$

This condition is equivalent to coming from the distances along a weighted tree which has negative weights along the internal edges [14, Sec. 4]. In this section, we will always take the deficiency graph to be with respect to the Plücker relations in (8).

As with the previous notions of rank, the tree rank of a 0/1 matrix can be characterized in terms of covers of graphs. For any disjoint subsets $I_1, \ldots, I_k \subset [n]$ (not necessarily a partition), the *complete k-partite graph* is the graph which has an edge between the elements of I_i and I_j for all $i \neq j$. Complete k-partite graphs are characterized by the property that among vertices which are incident to some edge, the relation of having a non-edge is a transitive relation.

Remark 6. The complete k-partite graphs defined above are exactly those graphs whose edge set forms the set of bases of a rank 2 matroid on n elements. The transitivity of being a non-edge is equivalent to the basis exchange axiom. Alternatively, each of the sets $I_1, \ldots, I_k$ partition the set of non-loops in the matroid into parallel

classes. See [11] for definitions of these terms. In the following proposition, we will see that the Plücker relations imply the basis exchange axiom for the 0 entries of a non-negative tree matrix.

Proposition 15. *Let M be an $n \times n$ $0/1$ dissimilarity matrix and let r be smallest size of a cover of G_M by complete k-partite subgraphs. As in Proposition 11, we only require every edge to be in the cover, not necessarily every vertex. If G_M has at most one isolated vertex then M has tree rank r. Otherwise, M has tree rank $r+1$.*

Proof. Let $I_1, \dots, I_k$ be disjoint sets defining a k-partite graph. We construct a tree which has $k+1$ internal vertices: one vertex v_i for each $i \in [k]$ and a vertex w. Every element of I_i has a branch of length $1/2$ to v_i and each v_i has a branch of length $-1/2$ to w. The elements of $J = [n] \setminus (I_1 \cup \cdots \cup I_k)$ are connected to w by a branch of length 1. This gives a distance matrix whose entries are 0 for the edges of the k-partite graph, 2 for the entries between elements of J, and 1 elsewhere. If G_M has at most one isolated vertex, then the sum of these matrices is M. Otherwise, adding the matrix with all entries entries equal to 1 yields M.

Conversely, suppose we have a decomposition of M as the sum of tree matrices. For each tree matrix, we can define a graph on the vertices $[n]$ with edges corresponding to the 0 entries. These form a cover of the graph of M, so we just need to show that the graph G_T coming from a tree matrix will be a complete k-partite graph. For this, we need to show that the relation of having a non-edge is a transitive relation among vertices which are incident to some edge. Suppose that (i,j) and (j,k) are non-edges, but (i,k) is an edge. Also suppose that j has an edge to some other vertex ℓ. Then $M_{ik} + M_{j\ell} = 0$, but M_{ij} and M_{jk} are positive and $M_{k\ell}$ and $M_{i\ell}$ are non-negative, which contradicts the Plücker relation. Thus, G_T is a complete k-partite graph, and we have proved the theorem when G_M has at most one isolated vertex.

Now suppose that G_M has at least 2 isolated vertices, i and j. There must be some tree matrix T in the decomposition of G_M such that $T_{ij} = 1$. Suppose that the corresponding graph G_T has an edge between two vertices k and ℓ, which must be distinct from i and j by assumption. Since i and j are isolated, T_{ik}, $T_{i\ell}$, T_{jk}, and $T_{j\ell}$ must each be at least 1. But $T_{ij} + T_{kl} = 1$, which contradicts the Plücker relation. Thus, G_T must be the trivial graph, so the decomposition of M must have one more term than a minimal cover of G_M. Therefore, M has rank $r+1$. □

Note that by taking the I_i in the definition of k-partite graph to be singletons, we get complete graphs, and by taking $k = 2$ with I_1 a singleton and I_2 any set disjoint from I_1, we get star trees. Together with Propositions 11 and 15, this confirms, for $0/1$-matrices, the second inequality in (2).

Lemma 16. *Let M be an $m \times m$ tree matrix, $n > m$ an integer, and C any real number. Then there exists an $n \times n$ tree matrix N, whose upper left $m \times m$ submatrix is M and such that the other entries are each at least C.*

Proof. The matrix M encodes the distances on some weighted tree T on m leaves. Pick any internal vertex v of T and let C' be the smallest distance between v and a leaf of m. Let T' be the tree on n leaves formed from T by attaching each leaf i with $m < i \leq n$ to v by an edge with weight $\max\{\frac{1}{2}C, C - C'\}$. Let N be the distance matrix of T' and N is a tree matrix with the desired properties. □

Proposition 17. *The dimension of the set of dissimilarity $n \times n$ matrices of tree rank at most r is the dimension of the classical secant variety,*

$$\begin{aligned} \binom{n}{2} - \binom{n-2r}{2} & \quad \text{if } r \leq \frac{n}{2}, \\ \binom{n}{2} & \quad \text{if } r \geq \frac{n-1}{2}. \end{aligned}$$

Proof. The tropical secant variety is contained in the tropicalization of the classical variety, which has the given dimension by [4, Thm. 2.1i]. Therefore, it is sufficient to prove that the tropical secant variety has at least the given dimension by the Bieri-Groves Theorem [2, Thm. A].

To prove the lower bound on the dimension, first note that for $r = \lfloor n/2 \rfloor$, $n-2r$ is either 0 or 1, so the first part of the statement implies that the dimension of the rth secant set is $\binom{n}{2}$, the dimension of the space of dissimilarity $n \times n$ matrices. Since higher secant varieties are at least as large as the preceeding ones, this implies the second part of the statement.

The proof of the first part, when $r \leq n/2$, is by induction on n. For $n \leq 3$ all dissimilarity matrices have tree rank 1, because the four-point condition is trivial. Thus, the dimension of the 1st secant set is $\binom{n}{2} = \binom{n}{2} - \binom{n-2}{2}$, as desired.

Now suppose $n > 3$ and by the inductive hypothesis, let N be an $(n-2) \times (n-2)$ matrix of tree rank at most $r-1$ such that the locus of matrices with tree rank at most $r-1$ has dimension $\binom{n-2}{2} - \binom{n-2r}{2}$ in a neighborhood of N.

Consider the caterpillar tree with leaves in the order 1, 3, 4, …, n, 2, pendant edge length p_i for leaf i, and negative internal edges $q_3, \dots, q_{n-1}$. Thus, the first two rows and columns of the distance matrix M are defined by:

$$\begin{aligned} M_{1,2} = M_{2,1} &= p_1 + q_3 + \cdots + q_{n-1} + p_2, \\ M_{1,i} = M_{i,1} &= p_1 + \sum_{j=3}^{i-1} q_j + p_i \quad \text{for } i > 2, \\ M_{2,i} = M_{i,2} &= p_i + \sum_{j=i}^{n-1} q_j + p_2 \quad \text{for } i > 2. \end{aligned}$$

Just from the values in these two rows, we can solve for the values of the edge lengths:

$$\begin{aligned} p_1 &= \frac{1}{2}(M_{1,2} + M_{1,3} - M_{2,3}) \\ p_2 &= \frac{1}{2}(M_{1,2} + M_{2,n} - M_{1,n}) \\ p_i &= \frac{1}{2}(M_{1,i} + M_{2,i} - M_{1,2}) \quad \text{for } i > 2 \\ q_i &= \frac{1}{2}(M_{1,i+1} + M_{2,i} - M_{1,i} - M_{2,i}) \end{aligned}$$

Therefore, the projection onto the first two columns has dimension $n + (n-3) = 2n - 3$ in a neighborhood of this point.

Assume that the p_i are sufficiently negative that the lower right $(n-2)\times(n-2)$ submatrix of M is less than N in every entry. Then

$$M \oplus \begin{bmatrix} * & \infty & \infty \\ \infty & * & \infty \\ \infty & \infty & N \end{bmatrix} = \begin{bmatrix} * & M_{12} & M_{13} & \cdots & M_{1n} \\ M_{12} & * & M_{23} & \cdots & M_{2n} \\ M_{13} & M_{23} & * & \cdots & N_{1,n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ M_{1n} & M_{2n} & N_{n-2,1} & \cdots & * \end{bmatrix}$$

has tree rank at most r and has a neighborhood of such matrices of dimension

$$2n-3+\binom{n-2}{2}-\binom{n-2r}{2}=\binom{n}{2}-\binom{n-2r}{2},$$

which is the desired expression. □

Unlike the cases of symmetric Barvinok rank and star tree rank, we do not know the maximum tree rank of a $n \times n$ dissimilarity matrix for large n. We have an upper bound of $n-2$ by Theorem 10, and we can improve on this slightly:

Theorem 18. *For $n \geq 6$, a $n \times n$ dissimilarity matrix M has tree rank at most $n-3$.*

Proof. Let M be a 6×6 dissimilarity matrix. Consider the tropical polynomial whose terms correspond to the perfect matchings on 6 vertices:

$$\begin{gathered} x_{12}x_{34}x_{56} \oplus x_{12}x_{35}x_{46} \oplus x_{12}x_{36}x_{45} \oplus x_{13}x_{24}x_{56} \oplus x_{13}x_{25}x_{46} \\ \oplus x_{13}x_{26}x_{45} \oplus x_{14}x_{23}x_{56} \oplus x_{14}x_{25}x_{36} \oplus x_{14}x_{26}x_{35} \oplus x_{15}x_{23}x_{46} \\ \oplus x_{15}x_{24}x_{36} \oplus x_{15}x_{26}x_{34} \oplus x_{16}x_{23}x_{45} \oplus x_{16}x_{24}x_{35} \oplus x_{16}x_{25}x_{34}. \end{gathered}$$

Note that this is the tropicalization of the Pfaffian of a 6×6 dissimilarity matrix (see, for example [10, Ch. 7]). After relabeling the vertices, we can assume that the minimum is achieved by the term $x_{12}x_{34}x_{56}$. In particular, this means that for the 4-point condition applied to the upper left 4×4 matrix, the minimum is achieved by $x_{12}x_{34}$. We can set X_1 equal to the smaller of $M_{13}+M_{24}-M_{12}$ and $M_{14}+M_{23}-M_{12}$, so that the matrix

$$N = \begin{bmatrix} * & M_{12} & M_{13} & M_{14} & \infty & \infty \\ M_{12} & * & M_{23} & M_{24} & \infty & \infty \\ M_{13} & M_{23} & * & X_1 & \infty & \infty \\ M_{14} & M_{24} & X_1 & * & \infty & \infty \\ \infty & \infty & \infty & \infty & * & \infty \\ \infty & \infty & \infty & \infty & \infty & * \end{bmatrix}$$

is a tree matrix. Moreover, by our assumption, we have that $X_1 \geq M_{34}$. By similar logic, there exist X_2 and X_3 such that the following is an expression of M as the tropical sum of 3 tree matrices:

$$N \oplus \begin{bmatrix} * & X_2 & \infty & \infty & M_{15} & M_{16} \\ X_2 & * & \infty & \infty & M_{25} & M_{26} \\ \infty & \infty & * & \infty & \infty & \infty \\ \infty & \infty & \infty & * & \infty & \infty \\ M_{15} & M_{25} & \infty & \infty & * & M_{56} \\ M_{16} & M_{26} & \infty & \infty & M_{56} & * \end{bmatrix} \oplus \begin{bmatrix} * & \infty & \infty & \infty & \infty & \infty \\ \infty & * & \infty & \infty & \infty & \infty \\ \infty & \infty & * & M_{34} & M_{35} & M_{36} \\ \infty & \infty & M_{34} & * & M_{45} & M_{46} \\ \infty & \infty & M_{35} & M_{45} & * & X_3 \\ \infty & \infty & M_{36} & M_{46} & X_3 & * \end{bmatrix}$$

Therefore, M has tree rank at most 3.

n	maximum tree rank	example
3	1	
4	2	
5	3	0/1 matrix corresponding to 5-cycle
6	3	
7	4	
8	5	
9	6	M in (9)
10	6 or 7	Any extension of M in (9)
$9k$	between $6k$ and $9k-3$	M_k from discussion following (9)

TABLE 1. Maximum possible tree rank of an $n \times n$ dissimilarity matrix, to the best of our knowledge. The upper bounds come from Theorems 10 and 18. The examples have the largest tree ranks that are known to us. The omitted examples can be provided by taking a principal submatrix of a larger example, by Lemma 19.

For $n > 6$, the theorem follows from Lemma 19. □

Beginning with $n = 10$, we don't know whether or not the bound in Theorem 18 is sharp. For the following 9×9 matrix, found by random search, the deficiency graph was computed to have chromatic number 6:

$$M = \begin{bmatrix} * & 1 & 6 & 7 & 2 & 3 & 8 & 9 & 6 \\ 1 & * & 2 & 7 & 9 & 7 & 5 & 7 & 1 \\ 6 & 2 & * & 6 & 0 & 6 & 1 & 7 & 1 \\ 7 & 7 & 6 & * & 3 & 3 & 8 & 5 & 3 \\ 2 & 9 & 0 & 3 & * & 5 & 7 & 5 & 7 \\ 3 & 7 & 6 & 3 & 5 & * & 9 & 3 & 9 \\ 8 & 5 & 1 & 8 & 7 & 9 & * & 2 & 3 \\ 9 & 7 & 7 & 5 & 5 & 3 & 2 & * & 8 \\ 6 & 1 & 1 & 3 & 7 & 9 & 3 & 8 & * \end{bmatrix} \tag{9}$$

Together with Theorem 18, this computation shows that M has tree rank 6. For any $k \geq 1$, we can form an $9k \times 9k$ matrix M_k by putting M in blocks along the diagonal and setting all other entries to 10. The deficiency graph of M_k includes k copies of the deficiency graph of M, and all edges between distinct copies. Therefore, the chromatic number, and thus the tree rank, are at least $6k$.

On the other hand, in order to provide examples of an $n \times n$ matrix with tree rank $n - 3$ for all $n \leq 9$, we have the following lemma.

Lemma 19. *Let M be an $n \times n$ matrix. If any $(n-m) \times (n-m)$ principal submatrix has tree rank r, then M has tree rank at most $r + m$.*

Proof. Fix a decomposition of the $(n-m) \times (n-m)$ principal submatrix into r tree matrices. We can extend each tree matrix to an $n \times n$ tree matrix by Lemma 16. For each index i not in the principal submatrix, define v_i to be the vector which is $C + M_{ij}$ in the jth entry and $-C$ in the ith entry, where C is a large real number. Then, the extended tree matrices, together with $\pi(v_i^T \odot v_i)$ for all i not in the principal submatrix, give a decomposition of M into $r + m$ tree matrices, as desired. □

These results on the maximum tree rank are summarized in Table 1.

6. Symmetric Barvinok rank for $n = 3$

In this section, we explicitly describe the stratification of 3×3 symmetric matrices by symmetric Barvinok rank. By Theorem 6, the rank is at most 3, and the locus of rank 1 is the tropical variety defined by the 2×2 minors, so it suffices to characterize the matrices of rank at most 2.

Following [7], we call a square matrix *tropically singular* if it lies in the tropical variety of the determinant.

Proposition 20. *Let M be a symmetric 3×3 matrix. Then the following are equivalent:*

(1) *M has symmetric Barvinok rank at most 2;*
(2) *The deficiency graph of M is 2-colorable;*
(3) *M is tropically singular and $M_{ii} + M_{jj} \leq 2M_{ij}$ for all $1 \leq i, j \leq 3$.*

Proof. First we note that $M_{ii} + M_{jj} \leq 2M_{ij}$ implies that every term of the tropical determinant is greater than or equal to $M_{11} + M_{22} + M_{33}$. Thus, a matrix M satisfying these inequalities is tropically singular if and only if some other term of the tropical determinant equals $M_{11} + M_{22} + M_{33}$.

Proposition 1 shows that (1) implies (2).

If the deficiency graph is 2-colorable, then it can't have any loops, so $M_{ii}+M_{jj} \leq 2M_{ij}$. Furthermore, the three diagonal entries can't form a clique, so without loss of generality, we assume that there is no edge between 11 and 22. Together with the inequality, this implies that $M_{11}+M_{22} = 2M_{12}$, so $M_{11}+M_{22}+M_{33} = 2M_{12}+M_{33}$, so by our initial remark, M is tropically singular. Therefore (2) implies (3).

Finally, suppose that M satisfies (3). We can subtract $M_{ii}/2$ from the ith row and ith column without changing the rank, and so we assume that every diagonal entry is 0. The inequalities then say that all of the off-diagonal entries are non-negative. For the minimum in the tropical determinant to be achieved at least twice, we must have at least one off-diagonal entry equal to 0. Without loss of generality, we assume that $M_{12} = 0$. Then,

$$M = \begin{bmatrix} 0 & 0 & M_{13} \\ 0 & 0 & M_{23} \\ M_{13} & M_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \infty \\ 0 & 0 & \infty \\ \infty & \infty & \infty \end{bmatrix} \oplus \begin{bmatrix} 2M_{13} & M_{13}+M_{23} & M_{13} \\ M_{13}+M_{23} & 2M_{23} & M_{23} \\ M_{13} & M_{23} & 0 \end{bmatrix},$$

where ∞ is as in Remark 1. Therefore, M has symmetric Barvinok rank at most 2. □

Remark 7. For larger matrices the symmetric Barvinok rank does not have as simple a characterization as the third condition in Proposition 20. A necessary condition for a symmetric $n \times n$ matrix to have rank at most r is that $M_{ii} + M_{jj} \leq 2M_{ij}$ and all the $(r+1)\times(r+1)$ submatrices are tropically singular, but this condition is not sufficient. For $n \geq 5$ and $r = n$, there are $n \times n$ symmetric matrices with finite rank greater than n by Remark 2. Even for $n = 4$ and for $r = 2$ and $r = 3$, the matrix

$$M = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix},$$

has symmetric Barvinok rank 4, as the nodes 12, 13, 24, and 34 form a 4-clique in its deficiency graph. However, M and all of its 3×3 submatrices are tropically singular.

7. Star tree rank for $n = 5$

In this section, we give an explicit characterization of the secant set of the space of star trees in the case $n = 5$. We do the same for the Grassmannian in the next section.

From Theorem 10, we know that the maximum star tree rank of a 5×5 matrix is 3. On the other hand, the set of dissimilarity matrices of star tree rank 1 are defined by the 2×2 minors. Thus, our task is to describe the second secant set of the space of star trees, i.e. the set of dissimilarity matrices of star tree rank 2.

First, we recall the defining ideal of the classical secant variety. The space of star trees is the tropicalization of the projection of the rank 1 symmetric matrices onto their off-diagonal entries. Its second secant variety is a hypersurface in $\mathbb{C}^{10}$ defined by the following 12-term quintic polynomial, known as the pentad [9]:

$$\begin{gathered} x_{12}x_{13}x_{24}x_{35}x_{45} - x_{12}x_{13}x_{25}x_{34}x_{45} - x_{12}x_{14}x_{23}x_{35}x_{45} + x_{12}x_{14}x_{25}x_{34}x_{35} \\ +x_{12}x_{15}x_{23}x_{34}x_{45} - x_{12}x_{15}x_{24}x_{34}x_{35} + x_{13}x_{14}x_{23}x_{25}x_{45} - x_{13}x_{14}x_{24}x_{25}x_{35} \\ - x_{13}x_{15}x_{23}x_{24}x_{45} + x_{13}x_{15}x_{24}x_{25}x_{34} - x_{14}x_{15}x_{23}x_{25}x_{34} + x_{14}x_{15}x_{23}x_{24}x_{35} \end{gathered}$$

Note that the 12 terms of the pentad correspond to the 12 different cycles on 5 vertices. The second secant set of the space of star trees is contained in the tropicalization of the pentad, but the containment is proper. Nonetheless, the terms of the pentad play a fundamental role in characterizing matrices of rank at most 2.

Theorem 21. *Let M be a 5×5 dissimilarity matrix. The following are equivalent:*

1. *M has star tree rank at most 2;*
2. *The deficiency graph of M is 2-colorable;*
3. *The minimum of the terms of the pentad is achieved at two terms which satisfy the following conditions:*
   - (a) *The terms differ by a transposition;*
   - (b) *Assuming, without loss of generality, that the minimized terms are $x_{12}x_{23}x_{34}x_{45}x_{15}$ and $x_{13}x_{23}x_{24}x_{45}x_{15}$, then we have that $M_{14}+M_{23} \leq M_{12} + M_{34} = M_{13} + M_{24}$.*

Proof. (1) implies (2) by Proposition 1.

We show that (2) implies (3) by proving the contrapositive. Suppose that M doesn't satisfy the two conditions for any pair of terms. Without loss of generality, we assume that $x_{12}x_{23}x_{34}x_{45}x_{15}$ is a minimal term from the pentad. By minimality, $M_{12} + M_{34}$ is less than or equal to $M_{13} + M_{24}$. If this inequality is strict, then we have an edge between 12 and 34 in the deficiency graph. On the other hand, if it is an equality, then by our assumption that the conditions in (3) don't hold, $M_{12}+M_{34}$ must be less than $M_{14}+M_{23}$, in which case we also have an edge between 12 and 34. Similarly, we have edges between 34 and 15, between 15 and 23, between 23 and 45, and between 45 and 12. Thus, the graph has a 5-cycle, and so is not 2-colorable. Therefore, M has star tree rank 3.

Finally, suppose that the two conditions in (3) hold. Let $A = M_{12} + M_{34} = M_{13} + M_{24}$. Then we claim that:

$$M = \begin{bmatrix} * & M_{12} & M_{13} & A - M_{23} & \infty \\ M_{12} & * & M_{23} & M_{24} & \infty \\ M_{13} & M_{23} & * & M_{34} & \infty \\ A - M_{23} & M_{24} & M_{34} & * & \infty \\ \infty & \infty & \infty & \infty & * \end{bmatrix} \oplus$$

$$\begin{bmatrix} * & M_{14}{+}M_{25}{-}M_{45} & M_{14}{+}M_{35}{-}M_{45} & M_{14} & M_{15} \\ M_{14}{+}M_{25}{-}M_{45} & * & B & M_{14}{+}M_{25}{-}M_{15} & M_{25} \\ M_{14}{+}M_{35}{-}M_{45} & B & * & M_{14}{+}M_{35}{-}M_{15} & M_{35} \\ M_{14} & M_{14}{+}M_{25}{-}M_{15} & M_{14}{+}M_{35}{-}M_{15} & * & M_{45} \\ M_{15} & M_{25} & M_{35} & M_{45} & * \end{bmatrix},$$

where ∞ is as in Lemma 9 and $B = M_{14} + M_{25} + M_{35} - M_{15} - M_{45}$. Note that each matrix has some entries taken from M, while the rest are forced by the rank 1 condition. We just need to check that the minimum of these two matrices is in fact M. To see this, we have the following inequalities from condition (3):

$$\begin{aligned} M_{14} + M_{23} \le M_{12} + M_{34} \quad &\Rightarrow \quad M_{14} \le A - M_{23} \\ M_{12} + M_{23} + M_{34} + M_{45} + M_{15} \;&\le\; M_{12} + M_{23} + M_{35} + M_{45} + M_{14} \\ &\Rightarrow \quad M_{34} \le M_{14} + M_{35} - M_{15} \\ M_{13} + M_{23} + M_{24} + M_{45} + M_{15} \;&\le\; M_{13} + M_{23} + M_{25} + M_{45} + M_{14} \\ &\Rightarrow \quad M_{24} \le M_{14} + M_{25} - M_{15} \\ M_{12} + M_{23} + M_{34} + M_{45} + M_{15} \;&\le\; M_{14} + M_{34} + M_{23} + M_{25} + M_{15} \\ &\Rightarrow \quad M_{12} \le M_{14} + M_{25} - M_{45} \\ M_{13} + M_{23} + M_{24} + M_{45} + M_{15} \;&\le\; M_{14} + M_{24} + M_{23} + M_{35} + M_{15} \\ &\Rightarrow \quad M_{13} \le M_{14} + M_{35} - M_{45} \\ M_{12} + M_{23} + M_{34} + M_{45} + M_{15} \;&\le\; M_{12} + M_{25} + M_{35} + M_{34} + M_{14} \\ &\Rightarrow \quad M_{23} \le B \end{aligned}$$

Therefore, M has star tree rank at most 2. □

8. Tree rank for $n = 5$

We now turn our attention to tree rank of 5×5 dissimilarity matrices. As in the previous section, the maximum tree rank is 3 and so it suffices to characterize 5×5 dissimilarity matrices of tree rank at most 2. Unlike the previous section, the second classical secant variety is already all of $\mathbb{C}^{10}$, so there is no classical polynomial whose tropicalization gives us a clue to the tropical secant set. However, the tropical pentad again shows up in our characterization.

First, here is a simple example of a 5×5 0/1 dissimilarity matrix with tree rank 3. Consider the 0/1 matrix corresponding to the 5-cycle C_5. Now, C_5 cannot be covered by fewer than 3 k-partite graphs, and so the matrix has tree rank at least 3 by Proposition 15. On the other hand, it has tree rank at most 3 by Theorem 10 and the inequality in (2). We will see in Remark 8 that this matrix is, in a certain sense, the only such example.

Let P be the tropical polynomial in variables $\{x_{ij} : 1 \le i < j \le 5\}$ which is the tropical sum of the 22 tropical monomials of degree 5 in which each $i \in$

$\{1,\ldots,5\}$ appears in a subscript exactly twice. Thus P has 12 monomials of the form $x_{12}x_{23}x_{34}x_{45}x_{15}$, forming the terms of the pentad, and 10 new monomials of the form $x_{12}x_{23}x_{31}x_{45}^2$. Let us call terms of the former kind *pentagons*, and terms of the latter kind *triangles*.

Theorem 22. *Let M be a 5×5 dissimilarity matrix. Then the following are equivalent:*

1. *M has tree rank at most 2;*
2. *The deficiency graph is 2-colorable;*
3. *The tropical polynomial P achieves its minimum at a triangle.*

Proof. First, (1) implies (2) by Proposition 1.

For (2) implies (3), we prove the contrapositive. Suppose the minimal terms of P are all pentagons; without loss of generality, we assume that $x_{12}x_{23}x_{34}x_{45}x_{15}$ is a minimal term. Since $x_{14}x_{45}x_{15}x_{23}^2$ is not minimal, we have $M_{12}+M_{34} < M_{14}+M_{23}$. Similarly, we have,

$$\begin{aligned}
&M_{12}+M_{23}+M_{34}+M_{45}+M_{15} < 2M_{15}+M_{23}+M_{34}+M_{24}, \text{ and}\\
&M_{12}+M_{23}+M_{34}+M_{45}+M_{15} < 2M_{45}+M_{12}+M_{23}+M_{13}.
\end{aligned}$$

Adding these together and cancelling, we get $M_{12}+M_{34} < M_{13}+M_{24}$. Thus, 12 and 34 are adjacent in the deficiency graph. By similar reasoning, we have adjacencies $12-34-15-23-45-12$ in the deficiency graph, so it has a five cycle and is not 2-colorable.

Finally, we prove that (3) implies (1). Assume without loss of generality that $x_{34}x_{35}x_{45}x_{12}^2$ is among the terms minimizing P. This implies that $x_{12}x_{34}$, $x_{12}x_{35}$, and $x_{12}x_{45}$ are each minimal terms in their respective Plücker equations. Then we can use Lemmas 23 and 24 below to obtain a decomposition of M into two tree matrices.

Lemma 23. *For any 5×5 dissimilarity matrix M such that $x_{12}x_{34}$, $x_{12}x_{35}$, and $x_{12}x_{45}$ are each minimal terms in their respective Plücker equations, there exists some 5×5 tree matrix T, such that for every $ij\in\binom{[5]}{2}$, we have $T_{ij}\geq M_{ij}$, with equality if $ij\in\{12,13,14,15,23,24,25\}$.*

Proof. If M satisfies

$$\begin{aligned}
M_{14}+M_{23} &\leq M_{13}+M_{24},\\
M_{15}+M_{24} &\leq M_{14}+M_{25},\\
M_{13}+M_{25} &\leq M_{15}+M_{23},
\end{aligned}$$

then adding shows that each inequality must be an equality. Thus, without loss of generality, we may assume that

$$M_{14}+M_{23} \geq M_{13}+M_{24}, \tag{10}$$

$$M_{15}+M_{24} \leq M_{14}+M_{25}, \tag{11}$$

$$M_{13}+M_{25} \leq M_{15}+M_{23}. \tag{12}$$

Every other case is equivalent to this one via permutations of $\{1,2\}$ and $\{3,4,5\}$.

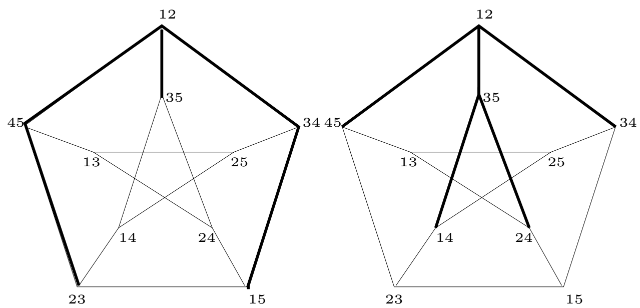

FIGURE 2. The two 2-colorable possibilities for Δ_M.

Now define T as follows. Let $T_{ij} = M_{ij}$ for $ij \in \{12, 13, 14, 15, 23, 24, 25\}$, and

$$\begin{aligned} T_{34} &= T_{24} + T_{13} - T_{12}, \\ T_{35} &= T_{25} + T_{13} - T_{12}, \\ T_{45} &= T_{15} + T_{24} - T_{12}. \end{aligned}$$

That T dominates M in every coordinate follows from the inequalities (10), (11), and (12). To check that T has tree rank 1, it suffices to check the four-point condition on each 4-tuple. This condition is satisfied for the 4-tuples $\{1, 2, 3, 4\}$, $\{1, 2, 3, 5\}$, and $\{1, 2, 4, 5\}$, by choice of T_{34}, T_{35}, T_{45}. Furthermore, we claim that

$$\begin{aligned} T_{15} + T_{34} = T_{13} + T_{45} &\leq T_{14} + T_{35}, \text{ and} \\ T_{24} + T_{35} = T_{25} + T_{34} &\leq T_{23} + T_{45}. \end{aligned}$$

These inequalities follow immediately from substituting for T_{34}, T_{35}, T_{45} and using inequalities (11) and (12). □

Lemma 24. *For any 5×5 dissimilarity matrix M, there exists some 5×5 tree matrix T' such that for every pair of indices i and j, we have $T'_{ij} \geq M_{ij}$, with equality if $ij \in \{34, 35, 45\}$.*

Proof. The $\{3, 4, 5\}$-principal submatrix of M is a tree matrix and therefore T' exists with the desired properties by Lemma 16. □

We can now finish the proof of Theorem 22. Let T be as given in Lemma 23 and let T' be as given in Lemma 24. Then $M = T \oplus T'$ and so M has tree rank at most 2. □

In the rest of this section, we present a detailed analysis of the deficiency graphs Δ_M for $n = 5$. There are 5 tropical Plücker relations on a 5×5 matrix, each containing 3 terms. Each term is the tropical product of terms with disjoint entries. Thus, Δ_M is a subgraph with at most 5 edges of the Petersen graph P_{10}, which is the graph on vertices $\binom{[5]}{2}$ with an edge between ij and kl if and only if $\{i, j\}$ and $\{k, l\}$ are disjoint sets. The following theorem describes the possible subgraphs that Δ_M can be.

Theorem 25. *Let M be a 5×5 dissimilarity matrix. Then the deficiency graph Δ_M is precisely one of the following:*

(1) *The trivial graph, in which case M has tree rank 1.*

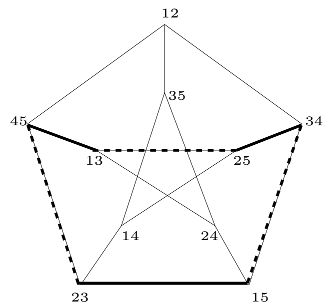

FIGURE 3. An alternating 6-cycle, where the solid edges lie in H and the dotted edges do not.

(2) *A non-trivial graph with fewer than* 5 *edges, in which case M has tree rank* 2*.*
(3) *Up to relabeling, either of the two graphs in Figure 2, in which case M has tree rank* 2*.*
(4) *A* 5*-cycle, in which case M has tree rank* 3*.*

Proof. The matrix M is a tree matrix if and only if the four-point condition holds for all 4-tuples, i.e. if and only if Δ_M is trivial. This is the first case.

Now suppose that Δ_M is a non-trivial graph with at most 4 edges. Then, at least one four-point condition holds, so Lemma 19 implies that M has tree rank at most 2. However, at least one four-point condition is violated, so M must have tree rank exactly 2.

In the rest of the proof, we assume that Δ_M has 5 edges, i.e. that each of the five 4-tuples yields one deficient pair. Thus, Δ_M contains exactly one of the three edges $12-34$, $13-24$, and $14-23$, and similarly for the remaining 4-tuples. Let us say that a subgraph H of P_{10} *admits an alternating even cycle* if there exists a cycle of even length in P_{10} such that, traversing the cycle, the edges are alternately members and nonmembers of H (Figure 3).

Lemma 26. *The graph Δ_M admits no alternating even cycle.*

Proof. Note that the only even cycles in the Petersen graph are of lengths 6 and 8 (see, for example, [3, Ch. 4.2]). Suppose C is an alternating cycle of length 6. All 6-cycles of P_{10} are isomorphic, so after relabeling, we may assume that C has vertices $45, 13, 25, 34, 15, 23$, in that order. Now suppose that $45-13$, $25-34$, and $15-23$ are the three edges of the cycle in H. Then

$$\begin{aligned} M_{45} + M_{13} &< M_{34} + M_{15}, \\ M_{25} + M_{34} &< M_{23} + M_{45}, \\ M_{15} + M_{23} &< M_{13} + M_{25}. \end{aligned}$$

But adding these inequalities yields a contradiction. If instead $13-25$, $34-15$, and $23-45$ are the three edges of the cycle lying in H, then we obtain a similar contradiction.

The case of a cycle of length 8 is analogous, so we omit it. □

We now use Lemma 26 to show that, up to relabeling, the only two possibilities for Δ_M, assuming that it is 2-colorable, are those in Figure 2. We organize our case analysis according to the maximum degree in the graph Δ_M.

If Δ_M has maximum degree 1, so that it is a perfect matching in P_{10}, then one may check that it has an alternating 8-cycle. This is impossible by Lemma 26.

If Δ_M has maximum degree 2, then it is either a 4-path and a 1-path, a 3-path and two 1-paths, or two 2-paths and a 1-path. (By a k-path we mean a path with k edges). It is easy to check that in each of these cases, Δ_M admits an alternating 6-cycle, which is again impossible by Lemma 26.

Thus, Δ_M has a vertex of degree 3, which we assume to be the vertex 12. In this case, one may check that, up to relabeling, Δ_M is one of the two graphs shown in Figure 2. Note that in either case, the graphs are 2-colorable, and thus M has tree rank 2 by Theorem 22.

Finally, if Δ_M is not 2-colorable, then it must have an odd cycle. The Petersen graph has no 3-cycles, so Δ_M must be a 5-cycle. □

Remark 8. If M is the 0/1 matrix corresponding to the 5-cycle C_5, then Δ_M is also a 5-cycle by Theorem 25. Explicitly, Δ_M has an edge for each non-adjacent pair of edges in the graph C_5. Moreover, Theorem 25 tells us that any other matrix N with tree rank 3 must have the same deficiency graph (up to relabeling). In this sense, M is essentially the only example of a 5×5 dissimilarity matrix with tree rank 3.

9. Open questions

- What is the maximum tree rank of a $n \times n$ dissimilarity matrix? Theorem 18 gives an upper bound, but beginning with $n = 10$, we do not know if it is sharp. Specifically, does there exist a 10×10 dissimilarity matrix with tree rank 7?
- Give a (reasonable) algorithm for computing tree rank. Note that both the tropical Veronese and the space of star trees are classical linear spaces, and so the results in [6] can be applied to compute star tree rank and symmetric Barvinok rank. However, it would be nice to have a good algorithm for computing tree rank.
- Is it true that the rank of a matrix, according to any of our notions, is always equal to the chromatic number of the corresponding deficiency graph? In Sections 6, 7, and 8, we observed that this was true for symmetric Barvinok rank with $n \leq 3$, star tree rank with $n \leq 5$, and tree rank with $n \leq 5$ respectively. In general, we believe the answer is no, but we do not know of a counterexample.
- In phylogenetics, trees have all edges, including the pendant edges, labeled by positive weights, or, equivalently, after negating, by negative weights. In this way, phylogenetic trees form a subset of the tropical Grassmannian, and we define the *phylogenetic tree rank* to be the rank with respect to this subset. One can ask the same questions about phylogenetic tree rank as in this paper: Which matrices have finite phylogenetic tree rank? What is the maximum possible finite phylogenetic tree rank? What is an explicit characterization of the secant sets for small matrices? Work in this direction was begun in [5].
- What is an algorithm for computing phylogenetic tree rank?

- Does a matrix have tree rank 2 if and only if all its principal 6×6 submatrices have tree rank at most 2? Pachter and Sturmfels have made the same conjecture for their definition of tree rank [12, p. 124].
- What is an explicit description of 6×6 matrices with tree rank at most 2, along the lines of Theorem 22? Does this help with the previous conjecture? There is a necessary condition coming from applying Theorem 22 to every 5×5 minor, and another from the fact that the matrix must be in the tropicalization of the classical secant variety. However, these two conditions together may not be sufficient for a 6×6 matrix to have tree rank at most 2.

Acknowledgments

We thank Bernd Sturmfels for his guidance and a close reading of the text. Maria Angélica Cueto helped us understand the connection to phylogenetics and William Slofstra pointed us to [10]. Melody Chan was supported by the Department of Defense through the National Defense Science and Engineering Graduate Fellowship Program.

References

[1] M. Akian, S. Gaubert, A. Guterman, Linear independence over tropical semirings and beyond, preprint, arXiv:0812.3496, 2008.
[2] R. Bieri, J. Groves, The geometry of the set of characters induced by valuations, J. Reine. Angew. Math. 347:168–195, 1984.
[3] J. A. Bondy, V. S. R. Murty, Graph Theory with Applications, Elsevier, New York, 1982.
[4] M. V. Catalisano, A. V. Geramita, A. Gimigliano, Secant varieties of Grassmann varieties, Proc. of the Amer. Math. Soc., 133:633–642, 2004. arXiv:math.AG/0208166.
[5] M. A. Cueto, Tropical mixtures of star tree metrics, preprint, arXiv:0907.2053, 2009.
[6] M. Develin, Tropical secant varieties of linear spaces, Discrete and Computational Geometry 35:117–129, 2006. arXiv:math.CO/0405115.
[7] M. Develin, F. Santos, B. Sturmfels, On the rank of a tropical matrix, in "Discrete and Computational Geometry" (E. Goodman, J. Pach and E. Welzl, eds), MSRI Publications, Cambridge University Press, 2005. arXiv:math.CO/0312114.
[8] J. Draisma, A tropical approach to secant dimensions, Journal of Pure and Applied Algebra, 212(2):349–363, 2008. arXiv:math.AG/0605345.
[9] M. Drton, B. Sturmfels, S. Sullivant, Algebraic factor analysis: tetrads, pentads and beyond, Probability Theory and Related Fields 138(3/4): 463–493, 2007. arXiv:0509.5390.
[10] C. D. Godsil, Algebraic Combinatorics, Chapman and Hall, New York, 1993.
[11] J. Oxley, Matroid Theory, Oxford Univ. Press, New York, 1992.
[12] L. Pachter, B. Sturmfels, Algebraic Statistics for Computational Biology, Cambridge University Press, Cambridge, 2005.
[13] S. Radziszowski, Small Ramsey Numbers, Electronic Journal of Combinatorics, Dynamic survey DS1. updated 2009.
[14] D. Speyer, B. Sturmfels, The tropical Grassmannian, Adv. Geom., 4(3):389–411, 2004. arXiv:math.AG/0304218.

Dept. of Mathematics, University of California, Berkeley, CA 94720, USA
E-mail address: `dustin@math.berkeley.edu, mtchan@math.berkeley.edu`